TWMS J. Pure Appl. Math. V.10, N.2, 2019, pp.239-245

ALGORITHM FOR SOLVING THE SYSTEMS OF THE GENERALIZED SYLVESTER-TRANSPOSE MATRIX EQUATIONS USING LMI

FIKRET A. ALIEV[1,2], VLADIMIR B. LARIN[3], NAILA VELIEVA[1], KAMILA GASIMOVA[4]
SHARGIYYA FARADJOVA[1]

Abstract. The solving for the systems of the generalized Sylvester-transpose matrix equations is given using the techniques of linear matrix inequalities (LMI). An algorithm for its solving is developed. The essence of the method is to minimize the linear functional, composed of the residual Sylvester equations and some unknown parameter. Examples of the analysis of the obtained results are provided.

Keywords: algebraic Sylvester equation, linear matrix inequality, linear systems, minimizing, residuals.

AMS Subject Classification: 15A06, 15A24, 15A69.

1. Introduction

The Sylvester equation

$$X - AXB = C,$$

the generalized Sylvester equation

$$EX - AXB = C,$$

and the linear matrix equations with more complex structure, for example, the system of the linear matrix equations

$$AXB + CYD = M,$$

$$EXF + GYH = N$$

attracted and continue to attract the attention of researchers (see, for example [1-6, 9-13, 15]). A biconjugate gradient algorithm (Bi-CG) for solving the systems of the generalized Sylvester-transpose matrix equations is suggested in [14]. Below, using the techniques of linear matrix inequalities (LMI) the solution of the systems of generalized Sylvester-transpose matrix equations is given.

[1]Institute of Applied Mathematics, Baku State University, Baku, Azerbaijan
[2]Institute of Information Technologies of ANAS, Baku, Azerbaijan
[3]Institute of Mechanics of the National Academy of Sciences of Ukraine, Ukraine
[4]Azerbaijan State Pedagogical University, Baku, Azerbaijan
f_aliev@yahoo.com, model@inmech.kiev.ua, nailavi@rambler.ru, kamile11@hotmail.com
Manuscript received July 2019.

2. Problem statement

The system of generalized Sylvester-transpose matrix equations [14] have the form

$$\begin{aligned} A_1PB_1 + C_1PD_1 + E_1P'F_1 = M_1, \\ A_2PB_2 + C_2PD_2 + E_2P'F_2 = M_2. \end{aligned} \tag{1}$$

It is assumed that in (1) the matrices

$$A_i, C_i \in R^{p_i\times m}, B_i, D_i \in R^{n\times q_i},\ E_i \in R^{p_i\times n},\ \ F_i \in R^{m\times q_i},\ M_i \in R^{p_i\times q_i}, i = 1, 2,$$

are known and have corresponding dimensions, $\prime$-means a transpose operation.

It is required to find the matrix $P \in R^{m\times n}$, which satisfies both equations (1) with a sufficiently high accuracy. Using the apparatus of linear matrix inequalities [7, 8] an algorithm was constructed for solving equations (1). On examples the results are compared with [14], with the accuracy of the solution of the equation differs by 10^{-4} and this shows the effectiveness of the proposed algorithm.

3. General relations for LMI

As noted in [7] (the relation (2.3), (2.4)), the matrix inequality:

$$\begin{bmatrix} Q(x) & S(x) \\ S'(x) & R(x) \end{bmatrix} > 0, \tag{2}$$

where the matrices $Q(x) = Q'(x)$, $R(x) = R'(x)$, $S(x)$ are linearly dependent on x, is equivalent to the following matrix inequalities:

$$R(x) > 0, Q(x) - S(x)R^{-1}(x)S'(x) > 0. \tag{3}$$

Let us consider the following LMI

$$\begin{bmatrix} Y & T \\ T' & I \end{bmatrix} > 0, Y = Y', \tag{4}$$

which according to (2), (3) can be written as

$$Y > TT'. \tag{5}$$

Hereinafter, I is the identity matrix of the corresponding size, T, Y- matrices of the corresponding dimensions to be determined.

Relations (4) can be generalized as the following system of LMI:

$$\begin{bmatrix} Y & T_i \\ T_i' & I \end{bmatrix} > 0, \quad Y = Y', \quad i = 1, 2. \tag{6}$$

which can be presented in a form similar to (5):

$$Y > T_iT_i', \ i = 1, 2. \tag{7}$$

With respect to (7), one can consider the standard LMI problem for eigenvalues, or the minimization problem

$$cx = tr(Y)$$

under the conditions (7), where $tr(Y)$ is the trace of the matrix Y. For solving this problem the procedure mincx.m in MATLAB package [7] is used.
Consider the algorithms for solving the equations (1). To do this, at first we will provide an explanation of the LMI for the relation (2) - (7).

4. The algorithm for solution of the equation (1)

Let us formulate solutions of the equation (1) in terms of LMI in the following form:
$P \geq 0,\ n > 0,\ Y > 0$ to be determined from the following inequalities

$$\begin{bmatrix} Y & (A_1PB_1 + C_1PD_1 + E_1P'F_1, -M_1) \\ (A_1PB_1 + C_1PD_1 + E_1P'F_1, -M_1)' & I \end{bmatrix} > 0, \quad (8)$$

$$\begin{bmatrix} Y & (A_2PB_2 + C_2PD_2 + E_2P'F_2, -M_2) \\ (A_2PB_2 + C_2PD_2 + E_2P'F_2, -M_2)' & I \end{bmatrix} > 0 \quad (9)$$

$$\begin{bmatrix} n & P \\ P' & I \end{bmatrix} > 0. \quad (10)$$

Thus, we can propose the following

Algorithm

(1) Based on the given matrices, the LMI is formed according to (8) - (10)

(2) Using the standard procedure mincx.m of the MATLAB package with condition (8) - (10), the n and Y are minimized (with the corresponding weight coefficients) and further, the required solution P is found.

According to the algorithm described above, the software was created in the MATLAB environment. Description of LMI operators in the MATLAB environment (with reference to Example 2, see the appendix).

Using computational procedure package MATLAB [7], the efficiency of the given algorithm is illustrated by the following examples.

5. Examples

Example 1. As a first example, consider the generalized Sylvester-transpose matrix equation

$$APB + CPD + EP'F = M. \quad (11)$$

Initial data taken from [14]

$$A = \begin{bmatrix} 4 & 40 & 4 & 7 & 9 & 1 & 0 & 10 \\ 4 & 400 & -99 & -2 & -2 & 2 & 3 & 4 \\ -2 & -2 & 100 & 5 & 600 & -1 & -5 & 5 \\ 100 & 2 & -2 & -2 & 5 & 1 & 200 & 2 \\ -90 & -9 & 10 & 5 & 200 & 3 & 1 & 3 \\ 10 & -20 & -1 & 50 & 4 & 5 & 3 & 10 \\ 20 & 3 & 900 & 6 & 3 & 5 & 9 & 4 \\ 20 & 3 & 233 & 6 & 3 & 5 & 9 & 4 \end{bmatrix},$$

$$B = \begin{bmatrix} 10 & -22 & 3 & 7 & 110 & -1 & 6 & 10 \\ 40 & -5 & 1 & -12 & 5 & 5 & 6 & 4 \\ -2 & 5 & 1 & 10 & 6 & -2 & 12 & 2 \\ 10 & 5 & -5 & -2 & 5 & -3 & 25 & 12 \\ 1 & -800 & 2 & 2 & 3 & 5 & 7 & 44 \\ 20 & -10 & -100 & 5 & 3 & 2 & 11 & 77 \\ 30 & 6 & 200 & 4 & 2 & 200 & 8 & 77 \\ 300 & 6 & 2 & 4 & 2 & 200 & 8 & 7700 \end{bmatrix},$$

$$\mathrm{C} = \begin{bmatrix} -32 & 168 & -4 & -14 & -422 & 6 & -24 & -20 \\ -152 & 820 & -202 & 44 & -24 & -16 & -18 & -8 \\ 4 & -24 & 196 & -30 & 1176 & 6 & -58 & 2 \\ 160 & -16 & 16 & 4 & -10 & 14 & 300 & -44 \\ -184 & 3182 & 12 & 2 & 388 & -14 & -26 & -170 \\ -60 & 0 & 398 & 80 & -4 & 2 & -38 & -288 \\ -80 & -18 & 1000 & -4 & -2 & -790 & -14 & -300 \\ -1160 & -18 & 458 & -4 & -2 & -790 & -14 & -3092 \end{bmatrix},$$

$$\mathrm{D} = \begin{bmatrix} -40 & -156 & -26 & -49 & -265 & -3 & -12 & -70 \\ -100 & -1990 & 493 & 34 & 0 & -20 & -27 & -28 \\ 14 & 0 & -502 & -45 & -3012 & 9 & 1 & -29 \\ -520 & -20 & 20 & 14 & -35 & 1 & -1050 & -34 \\ 448 & 1645 & -54 & -29 & -1006 & -25 & -19 & -103 \\ -90 & 120 & 205 & -260 & -26 & -29 & -37 & -204 \\ -160 & -27 & -4900 & -38 & -19 & -425 & -61 & -174 \\ -700 & -27 & -1169 & -38 & -19 & -425 & -61 & -15420 \end{bmatrix},$$

$$\mathrm{F} = \begin{bmatrix} -30 & -178 & -23 & -42 & -155 & -155 & -6 & -60 \\ -60 & -1995 & 494 & 22 & 5 & -15 & -21 & -24 \\ 12 & 5 & -501 & -35 & -3006 & 7 & 13 & -27 \\ -510 & -15 & 15 & 12 & -30 & -2 & -1025 & -22 \\ 449 & 845 & -52 & -27 & -1003 & -20 & -12 & -59 \\ -70 & 110 & 105 & -225 & -23 & -27 & -26 & -127 \\ -130 & -21 & -4700 & -34 & -17 & -225 & -53 & -97 \\ -400 & -21 & -1167 & -34 & -17 & -225 & -53 & -7720 \end{bmatrix},$$

$$\mathrm{E} = \begin{bmatrix} -28 & 208 & 0 & -7 & -413 & 7 & -24 & -10 \\ -148 & 1220 & -301 & 42 & -26 & -14 & -15 & -4 \\ 2 & -26 & 296 & -25 & 1776 & 5 & -63 & 7 \\ 260 & -14 & 14 & 2 & -5 & 15 & 500 & -42 \\ -274 & 3173 & 22 & 7 & 588 & -11 & -25 & -167 \\ -50 & -20 & 397 & 130 & 0 & 7 & -35 & -278 \\ -60 & -15 & 1900 & 2 & 1 & -785 & -5 & -296 \\ -1140 & -15 & 691 & 2 & 1 & -785 & -5 & -3078 \end{bmatrix}.$$

M=rand(8) –elements random matrix.

Solving the equation (1) with these initial data is obtained

$$P = \begin{bmatrix} 0.0050 & -0.0002 & -0.0007 & 0.0001 & 0.0002 & -0.0014 & 0.0001 & -0.0000 \\ -0.0003 & 0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0001 & -0.0000 & 0.0000 \\ -0.0002 & -0.0000 & 0.0000 & -0.0001 & -0.0000 & -0.0000 & 0.0000 & -0.0000 \\ 0.0007 & 0.0001 & -0.0000 & 0.0002 & 0.0000 & 0.0003 & -0.0003 & -0.0001 \\ 0.0001 & -0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.0006 & -0.0000 & 0.0002 & -0.0005 & -0.0000 & 0.0002 & 0.0000 & 0.0000 \\ -0.0012 & -0.0001 & 0.0001 & -0.0003 & -0.0000 & 0.0003 & 0.0004 & 0.0000 \\ -0.0001 & -0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}. \tag{12}$$

The solution (12) satisfies equation (11) with an accuracy of 10^{-12}, which in [14] is 10^{-8} (see Figure 2 [14]).

The obtained result demonstrates the efficiency of the given algorithm.

Example 2. Let the coefficients of equation (1) be given in the following form

$$A_1 = [3\ 4\ 5]\,, B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; C_1 = \begin{bmatrix} 7 & 9 & 8 \end{bmatrix},\ D_1 = 3 \times B_1,\ E_1 = A_1/3,\ F_1 = D_1/5,$$

$$A_2 = [5\ 3\ 7]\,, B_2 = \begin{bmatrix} 1 & 5 & 4 \\ 2 & 3 & 1 \\ 6 & 7 & 8 \end{bmatrix}; C_2 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix},\ D_2 = 2 \times B_2,\ E_2 = A_2/5,\ F_2 = D_2/3,$$

$$X = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

This value X is then used for calculating the right sides of equations (1) (matrix M_1, M_2). Solving these equations using the above algorithm, we obtain the solution (1) the norm is less than the norm X.

The calculated residuals of the systems are in the form:
nev1 = 1.7053e-014; nevf1=1.7053e-014; nev2=5.0842e-013; nevf2=5.0842e-013
where nev1 and nev2 are the residuals of the first and second equations (1), nevf1 and nevf2 are the Frobenius norms for the first and second equations (1), respectively.

6. Appendix

The description of the operators in the mincx procedure in the MATLAB environment is:

```
[mb1,mb2]=size(B1);
setlmis([]);
p=lmivar(2,[3 3]);
y=lmivar(1,[1 1]);
n=lmivar(1,[mb1 1]);
lmiterm([-1 1 1 y],1,1);
lmiterm([1 1 2 0],-M1);
lmiterm([1 1 2 p],A1,B1);
lmiterm([1 1 2 p],C1,D1);
lmiterm([1 1 2 -p],E1,F1);
```

```
lmiterm([-1 2 2 0],1);
lmiterm([-2 1 1 y],1,1);
lmiterm([2 1 2 0],-M2);
lmiterm([2 1 2 p],A2,B2);
lmiterm([2 1 2 p],C2,D2);
lmiterm([2 1 2 -p],E2,F2);
lmiterm([-2 2 2 0],1);
lmiterm([-3 1 1 n],1,1);
lmiterm([3 1 2 p],1,1);
%lmiterm([-3 1 2 0],Xf);
lmiterm([-3 2 2 0],1);
lmiterm([-3 1 1 y],1,1);
%lmiterm([-4 1 1 n],1,1);
lmis=getlmis;
%[alpha,popt]=gevp(lmis,1);
nn=decnbr(lmis);
c=zeros(nn,1);
for j=1:nn,
[yj]=defcx(lmis,j,y);
c(j)=trace(yj);end;
c1=zeros(nn,1);
for j=1:nn,
[xj]=defcx(lmis,j,n);
c1(j)=trace(xj);end;
%c,c1,pause;
c=c+1e-16*c1;
options=[1e-10,0,0,0,0];
[copt,popt]=mincx(lmis,c,options);
P=dec2mat(lmis,popt,p),
nev1=norm(M1-(A1*P*B1+C1*P*D1+E1*P'*F1)),
nevf1=norm(M1-(A1*P*B1+C1*P*D1+E1*P'*F1),'fro'),
nev2=norm(M2-(A2*P*B2+C2*P*D2+E2*P'*F2)),
nevf2=norm(M2-(A2*P*B2+C2*P*D2+E2*P'*F2),'fro'),
np=norm(P),nx=norm(X),
```

7. Conclusion

For solving the generalized Sylvester-transpose matrix equations the algorithms are proposed based on computational procedures for linear matrix inequalities in MATLAB environment. The effectiveness of results is shown with several examples.

References

[1] Aliev, F.A., Larin, V.B., (2019), On solution of modified matrix Sylvester equation, TWMS Journal of Applied and Engineering Mathematics, 9(3), pp.549-553.

[2] Aliev, F.A., Larin, V.B., (2019), On the solving of matrix equation of Sylvester type, Computational Methods for Differential Equations, 7(1), pp.96-104.

[3] Aliev, F.A., Larin, V.B., (2015), Algorithm for solving of the generalized Sylvester equation, 5-th International Conference on Control and Optimization with Industrial Applications, 27-29 August, Baku, Azerbaijan, pp.41-43.

[4] Aliev, F.A., Larin, V.B., (2017), About solution of constrained matrix Sylvester equation, Proceedings of IAM, 6(1), pp.102-108.

[5] Aliev, F.A., Larin, V.B., (2018), Solving the system of Sylvester matrix equations, International Journal of Applied Mechanics, 54(5), pp.611-616.

[6] Aliev, F.A., Larin, V.B., Velieva, N.I., Gasimova, K.G., (2017), On periodic solution of generalized Sylvester matrix equations, Appl. Comput. Math., 16(1), pp.87-101

[7] Boyd, S., Ghaoui, L.E., Feron, E., Balakrishnan, V., (1994), Linear Matrix Inequalities in System and Control Theory, Philadelphia: SIAM, 289p.

[8] Gahinet, P., Nemirovski, A., Laub, A.J., Chilali, M., (1995), LMI Control Toolbox Users Guide, The Math Works Inc., 306 p.

[9] Hajarian, M., (2015), Developing CGNE algorithm for the periodic discrete-time generalized coupled Sylvester matrix equations, Comp. Appl. Math., 34, pp.755-771.

[10] Larin, V.B., (2009), On solution of Sylvester equation, J. of Automation and Information Sciences, 41(1), pp.1-7.

[11] Larin, V.B., (2016), On solution of the generalized Riccati equations, Problems of Control and Informatics, 6, pp.5-9.

[12] Larin, V.B., (2015), On solution of the linear matrix equations, J. of Automation and Information Sciences, 47(9), pp.1-9.

[13] Larin, V.B., Aliev, F.A., (2008), On the use of Bass relation for solving of matrix equations, Rep. of NAS Azerbaijan, 4, pp.15-25.

[14] Masoud, Hajariana, (2008), Biconjugate residual algorithm for solving general Sylvester-transpose matrix equations, Filomat, 32(15), pp.5307-5318, https://doi.org/10.2298/FIL1815307H.

[15] Pourgholi, R., Esfahani, A., Houlari, T., Foadian, S., (2017), An application of Sinc-Galerkin method for solving the Tzou equation, Appl. Comput. Math., 16(3), pp.240-256.

Fikret A. Aliev, for a photograph and biography, see TWMS J. Pure Appl. Math., V.1, N.1, 2010, p.12.

Vladimir B. Larin, for a photograph and biography, see TWMS J. Pure Appl. Math., V.2, N.1, 2011, p.160.

Naila Velieva, for a photograph and biography, see TWMS J. Pure Appl. Math., V.4, N.1, 2013, p. 109.

Kamila Gasimova was born in 1968. She works at the Department of Computational Mathematics and Computer Science of Azerbaijan State Pedagogical University. Her research interests are computational mathematics, informatics and discreet systems.

Shargiyya Faradjova, was born in 1991 in Baku, Azerbaijan. She graduated from Baku State University, Applied Mathematics and Cybernetics Faculty. From 2014 she is a researcher at the Institute of Applied Mathematics. Her research interests are spectral analysis, differential and algebraic equations, control theory.